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A Perturbation Result for a Double Eigenvalue Hemivariational Inequality with Constraints and Applications

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Abstract. In this paper we prove a perturbation result for a new type of eigenvalue problem introduced by D. Motreanu and P.D. Panagiotopoulos (1998). The perturbation is made in the nonsmooth and nonconvex term of a double eigenvalue problem on a spherlike type manifold considered in 'Multiple solutions for a double eigenvalue hemivariational inequality on a spherelike type manifold' (to appear in *Nonlinear Analysis*). For our aim we use some techniques related to the Lusternik-Schnirelman theory (including Krasnoselski's genus) and results proved by J.N. Corvellec et al. (1993), M. Degiovanni and S. Lancelotti (1995), and V.D. Rădulescu and P.D. Panagiotopoulos (1998). We apply these results in the study of some problems arising in Nonsmooth Mechanics.

Key words: Double eigenvalue hemivariational inequality, Multiplicity result, Nonconvex perturbation, Coupled semilinear Poisson equation, Adhesively connected von Kármán plates

1. Introduction

The mathematical theory of hemivariational inequalities and their applications in mechanics, engineering or economics, were introduced and developed by P.D. Panagiotopoulos [17-23]. This theory may be considered as an extension of the theory of variational inequalities studied by G. Fichera [6], J.L. Lions and G. Stampacchia [8]. However, Hemivariational Inequalities are much more general, in the sense that they are not equivalent to minimum problems, but give rise to substationarity problems.

In this paper we deal with a new type of eigenvalue problem for hemivariational inequalities, called 'double eigenvalue problems' which were introduced by D. Motreanu and P.D. Panagiotopoulos [9]. By M.F. Bocea, D. Motreanu and P.D. Pangiotopoulos [1] it is proved a multiplicity result concerning the solutions belonging to a spherelike type manifold. Our aim is to study the effect induced by an arbitrary perturbation made in the nonsmooth and nonconvex term of the symmetric hemivariational inequality considered in [1].

2. The abstract framework

Let V be a real Hilbert space, with the scalar product and the associated norm denoted by $(\cdot,\cdot)_V$ and $\|\cdot\|_V$, respectively. We shall suppose that V is densely and compactly embedded in $L^p(\Omega;\mathbf{R}^N)$ for some $p\geqslant 2$, where $N\geqslant 1$ and $\Omega\subset\mathbf{R}^m$, $m\geqslant 1$, is a smooth, bounded domain. Throughout in this paper, we shall denote by $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle$ the duality products on V and $\mathbf{R}^N$, respectively. Let us denote by $C_p(\Omega)$ the constant of the (continuous, in particular) embedding $V\subset L^p(\Omega;\mathbf{R}^N)$ which means that

$$\|v\|_{L^p}\leqslant C_p(\Omega)\cdot\|v\|_V,\ \text{for all } v\in V.$$

Let $a_1, a_2 : V\times V\to\mathbf{R}$ be two continuous symmetric bilinear forms on V and let $B_1, B_2 : V\to V$ be two bounded self-adjoint linear operators which are coercive in the sense that

$$(B_i v, v)_V\geqslant b_i\cdot\|v\|_V^2,\ \text{for all } v\in V,\ i=1,2,$$

for some constants $b_1, b_2>0$. For fixed positive numbers a, b, r we consider the submanifold $S_r^{a,b}$ of $V\times V$ described as follows

$$S_r^{a,b}=\{(v_1,v_2)\in V\times V\ :\ a(B_1v_1,v_1)_V+b(B_2v_2,v_2)_V=r^2\}.$$

We need to consider the tangent space associated to the manifold defined above, which is

$$T_{(u_1,u_2)}S_{a,b}^r:=\{(v_1,v_2)\in V\times V\ :\ a(B_1u_1,v_1)_V+b(B_2u_2,v_2)_V=0\}.$$

Let $j:\Omega\times\mathbf{R}^N\to\mathbf{R}$ satisfy the following assumptions

(i) $j(\cdot,y)$ is measurable in Ω for each $y\in\mathbf{R}^N$ and $j(\cdot,0)$ is essentially bounded in Ω;
(ii) $j(x,\cdot)$ is locally Lipschitz in $\mathbf{R}^N$ for a.e. $x\in\Omega$.

Throughout this paper we shall use the notation j_y^0 for Clarke's generalized directional derivative (see [3]) of j with respect to the second variable y, i.e.,

$$j_y^0(x,y;z)=\limsup_{\substack{w\to y\\ \lambda\downarrow 0}}\frac{j(x,w+\lambda z)-j(x,w)}{\lambda},$$

with $x\in\Omega$, $y,z\in\mathbf{R}^N$ and $\lambda\in\mathbf{R}$. Accordingly, Clarke's generalized gradient $\partial_y j(x,y)$ of the locally Lipschitz map $j(x,\cdot)$ is defined by

$$\partial_y j(x,y)=\{\xi\in\mathbf{R}^N\ :\ \langle\xi,z\rangle\leqslant j_y^0(x,y;z),\ \forall z\in\mathbf{R}^N\}.$$

As Rădulescu and Panagiotopoulos observed in [24], we may request that j satisfies a slight more general growth condition than the classical one (see the hypothesis (H_1) in Motreanu and Panagiotopoulos [13])

(H_1) There exist $\theta \in L^{\frac{p}{(p-1)}}(\Omega)$ and $\rho \in \mathbf{R}$ such that

$$|z| \leqslant \theta(x) + \rho|y|^{p-1}, \tag{1}$$

for a.e. $(x, y) \in \Omega \times \mathbf{R}^N$ and each $z \in \partial_y j(x, y)$.

Let us consider a real function $C : S_r^{a,b} \times V \times V \to \mathbf{R}$ to which we impose no continuity assumption. We are now prepared to consider the following double eigenvalue problem : Find $u_1, u_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbf{R}$ such that

$$(P^1_{r,a,b}) \begin{cases} a_1(u_1, v_1) + a_2(u_2, v_2) + C((u_1, u_2), v_1, v_2) + \\ + \int_\Omega j_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x))dx \geqslant \\ \geqslant \lambda_1(B_1u_1, v_1)_V + \lambda_2(B_2u_2, v_2)_V, \ \forall\, v_1, v_2 \in V, \\ \\ a(B_1u_1, u_1)_V + b(B_2u_2, u_2)_V = r^2. \end{cases}$$

We impose the following hypothesis

(H_2) There exist two locally Lipschitz maps $f_i : V \to \mathbf{R}$, bounded on $\pi_i(S_r^{a,b})$, $(i = 1, 2)$ respectively, and such that the following inequality holds

$$C((u_1, u_2), v_1, v_2) \geqslant f_1^0(u_1; v_1) + f_2^0(u_2; v_2), \tag{2}$$
$$\forall\ (u_1, u_2) \in S_r^{a,b} \text{ and } \forall\ (v_1, v_2) \in T_{(u_1,u_2)}S_r^{a,b}.$$

In addition we suppose that the sets

$$\{z \in V^* \ :\ z \in \partial f_i(u_i), u_i \in \pi_i(S_r^{a,b})\}$$

are relatively compact in V^*, for $i = 1, 2$.

Define the map $(A_1, A_2) : V \times V \to V^* \times V^*$ by the relation

$$\langle (A_1, A_2)(u_1, u_2), (v_1, v_2)\rangle_{V\times V} = a_1(u_1, v_1) + a_2(u_2, v_2) \tag{3}$$

and the duality map $\Lambda : V \times V \to V^* \times V^*$ given by

$$\langle \Lambda(u_1, u_2), (v_1, v_2)\rangle_{V\times V} = a(B_1u_1, v_1)_V + b(B_2u_2, v_2)_V. \tag{4}$$

We also assume

(H_3) For every sequence $\{(u_n^1, u_n^2)\} \subset S_r^{a,b}$ with $u_n^i \rightharpoonup u_i$ weakly in V, for any $z_n^i \in \partial f_i(u_n^i)$, with

$$a_i(u_n^i, u_n^i) + \langle z_n^i, u_n^i\rangle_V \to \alpha_i \in \mathbf{R}, \tag{5}$$

$i = 1, 2$, and for all $w \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ which satisfies the relation

$$w(x) \in \partial_y j(x, (u_1 - u_2)(x)) \text{ for a.e. } x \in \Omega, \tag{6}$$

such that

$$[(A_1, A_2) - \lambda_0 \cdot \Lambda]\,(u_n^1, u_n^2)$$

converges in $V^* \times V^*$, where

$$\lambda_0 = r^{-2}(\alpha_1 + \alpha_2 + \int_\Omega \langle w(x), (u_1 - u_2)(x)\rangle \, dx), \tag{7}$$

there exists a convergent subsequence of (u_n^1, u_n^2) in $V \times V$ (thus, in $S_r^{a,b}$).

(H_4) j is even with respect to the second variable, i.e.,

$$j(x, -y) = j(x, y), \text{ for a.e. } x \in \Omega, \text{ and any } y \in \mathbf{R}^N,$$

and f_i is even on $\pi_i(S_r^{a,b})$ i.e.,

$$f_i(-u_i) = f_i(u_i), \text{ for all } (u_1, u_2) \in S_r^{a,b}, i = 1, 2.$$

By assuming the hypotheses $(H_1), (H_2), (H_3)$ and (H_4), it is proved in [1] that the double eigenvalue problem $(P^1_{r,a,b})$ admits infinitely many pairs of solutions $\{\pm(u_n^1, u_n^2), (\lambda_n^1, \lambda_n^2)\} \subset S_r^{a,b} \times \mathbf{R}^2$. Moreover, it is found the expression of the eigenvalues λ_n^1 and λ_n^2. The aim of this paper is to answer a natural question: what happens if we perturb $(P^1_{r,a,b})$ in a suitable manner? For proving our main result we need some notions of Algebraic Topology which may be found in Spanier [26]. We recall now only some basic definitions.

Let X be a metric space and $A \subset X$. We said that a map $r : X \to A$ is a *retraction* if it is continuous, surjective and fulfills $r_{|A} = Id$. A retraction r is called to be a *strong deformation retraction* if there exists a homotopy $F : X \times [0, 1] \to X$ of $i \circ r$ and Id_X such that $F(x, t) = F(x, 0)$, for each $(x, t) \in A \times [0, 1]$. Here i stands for the inclusion map of A in X. We call X to be *weakly locally contractible*, if every point has a contractible neighbourhood in X. Let $\xi : X \to \mathbf{R}$ be a locally Lipschitz functional. Set, for every $a \in \mathbf{R}$

$$[\xi \leqslant a] := \{u \in X; \xi(u) \leqslant a\}.$$

Let us fix $a, b \in \mathbf{R}$ with $a \leqslant b$. The pair $([\xi \leqslant b], [\xi \leqslant a])$ is called *trivial* if, for every neighbourhoods $[a^{'}, a^{''}]$ of a and $[b^{'}, b^{''}]$ of b, there exist some closed sets A and B such that $[\xi \leqslant a^{'}] \subset A \subset [\xi \leqslant a^{''}]$, $[\xi \leqslant b^{'}] \subset B \subset [\xi \leqslant b^{''}]$ and such that A is a strong deformation retract of B.

The next notion is essentialy due to M. Degiovanni and S. Lancelotti [5].

A real number c is said to be an *essential value* of ξ if, for every $\epsilon > 0$, there exist $a, b \in (c - \epsilon, c + \epsilon)$, with $a < b$ and such that the pair $([\xi \leqslant b], [\xi \leqslant a])$ is not trivial.

Let us consider an arbitrary element ϕ in V^* and $g : \Omega \times \mathbf{R}^N \to \mathbf{R}$ a Caratheodory function which is locally Lipschitz with respect to the second variable and such that $g(\cdot, 0) \in L^1(\Omega)$. Let us consider the following non-symmetric perturbed double eigenvalue problem: find $(u_1, u_2) \in V \times V$ and $(\lambda_1, \lambda_2) \in \mathbf{R}^2$ such that

$$(P^2_{r,a,b}) \begin{cases} a_1(u_1, v_1) + a_2(u_2, v_2) + C((u_1, u_2), v_1, v_2) + \\ + \int_\Omega \{ j^0_y(x, (u_1 - u_2)(x); (v_1 - v_2)(x)) + \\ g^0_y(x, (u_1 - u_2)(x); (v_1 - v_2)(x))\} dx + \\ + < \phi, v_1 >_V + < \phi, v_2 >_V \geqslant \\ \geqslant \lambda_1 (B_1 u_1, v_1)_V + \lambda_2 (B_2 u_2, v_2)_V, \ \forall\, v_1, v_2 \in V, \\ \\ a(B_1 u_1, u_1)_V + b(B_2 u_2, u_2)_V = r^2. \end{cases}$$

Fix $\delta > 0$. We impose to g the growth condition

(H_5) There exist $\theta_1 \in L^{\frac{p}{(p-1)}}(\Omega)$ and $\delta > 0$ such that

$$|z| \leqslant \theta_1(x) + \delta |y|^{p-1}, \tag{8}$$

for a.e. $(x, y) \in \Omega \times \mathbf{R}^N$ and each $z \in \partial_y g(x, y)$.

Let us denote by J and G the (locally Lipschitz, by hypotheses (H_1) and (H_5)) functionals from $L^p(\Omega; \mathbf{R}^N)$ into $\mathbf{R}$, defined by

$$J(u) = \int_\Omega j(x, u(x)) dx \text{ and } G(u) = \int_\Omega g(x, u(x)) dx.$$

We associate to the problems $(P^1_{r,a,b})$ and $(P^2_{r,a,b})$ the energy functions $I_1, I_2 : V \times V \to \mathbf{R}$, defined by

$$I_1(u_1, u_2) = \frac{1}{2} \cdot [a_1(u_1, u_1) + a_2(u_2, u_2)] + \\ + f_1(u_1) + f_2(u_2) + J(u_1 - u_2), \tag{9}$$

and

$$I_2(u_1, u_2) = I_1(u_1, u_2) + G(u_1 - u_2) + \langle \phi, u_1 \rangle_V + \langle \phi, u_2 \rangle_V, \tag{10}$$

for all $u_1, u_2 \in V$.

We denote by Υ the family of closed and symmetric with respect to the origin $0_{V\times V}$, subsets of $S^{a,b}_r$. Let us denote, as usually, by $\gamma(S)$ the Krasnoselski's genus of the set $S \in \Upsilon$, that is, the smallest integer $k \in \mathbf{N} \cup \{+\infty\}$ for which there exists an odd continuous mapping from S into $\mathbf{R}^k \backslash \{0\}$. For every $n \geqslant 1$, set

$$\Gamma_n = \{ S \subset S^{a,b}_r \ : \ S \in \Upsilon, \gamma(S) \geqslant n \}$$

Recall that the corresponding minimax values of I_1 over Γ_n

$$\beta_n = \inf_{S \subset \Gamma_n} \sup_{(u_1, u_2) \in S} \{ I_1(u_1, u_2) \},$$

are critical values of I_1 on $S^{a,b}_r$ (see [1, Theorem 1]).

3. Preliminary results

The first result of this section concerns the functional I_1.

LEMMA 1. *Let* $s := \sup_{(u_1,u_2)\in S_r^{a,b}}\{I_1(u_1, u_2)\}$. *Then the supremum is not achieved and* $\lim_{n\to\infty}\beta_n = s$. *Moreover, there exists a sequence* (b_n) *of essential values of the restriction of* I_1 *at* $S_r^{a,b}$, *strictly increasing to* s.

Proof. This result is essentially proved in [24] (see Lemma 1) by using the ideas of M. Degiovanni and S. Lancelotti (see [5], Theorem 2.12). The only difference is that now, we work not on a sphere but on the Riemannian manifold $S_r^{a,b}$. It is sufficient to point out that this is a weakly locally contractible space as the usual sphere in V is, and the fact that I_1 satisfies the Palais-Smale condition on $S_r^{a,b}$ as was proved in [1]. With these remarks, the proof of the Lemma 1 follows the same steps with the one in [24]. $\square$

For continuing, we need two aditional assumptions

(H_6) The following inequalities hold

$$\|\theta_1\|_{L^{\frac{p}{p-1}}} \leqslant \delta,\ \ \|g(\cdot,0)\|_{L^1} \leqslant \delta \text{ and } \|\phi\|_{V^*} \leqslant \delta. \tag{11}$$

The second assumption is actually a variant of the compactness hypothesis (H_3)

(H_7) For every sequence $\{(u_n^1, u_n^2)\} \subset S_r^{a,b}$ with $u_n^i \rightharpoonup u_i$ weakly in V, for any $z_n^i \in \partial f_i(u_n^i)$, with

$$a_i(u_n^i, u_n^i) + \langle z_n^i, u_n^i\rangle_V + < \phi, u_n^i >_V \to \alpha_i \in \mathbf{R}, \tag{12}$$

$i = 1, 2$ and for all $w, z \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ which satisfies the relations

$$\begin{aligned} &w(x) \in \partial_y j(x, (u_1 - u_2)(x)),\\ &z(x) \in \partial_y g(x, (u_1 - u_2)(x)),\ \text{for a.e. } x \in \Omega, \end{aligned} \tag{13}$$

such that

$$[(A_1, A_2) - \lambda_0 \cdot \Lambda]\,(u_n^1, u_n^2)$$

converges in $V^* \times V^*$, where,

$$\lambda_0 = r^{-2}(\alpha_1 + \alpha_2 + \int_\Omega \langle w(x) + z(x), (u_1 - u_2)(x)\rangle\, dx), \tag{14}$$

there exists a convergent subsequence of (u_n^1, u_n^2) in $V \times V$.

The next result proves that if $\delta > 0$ is sufficiently small in the hypotheses (H_5) and (H_6), then I_2 is a small perturbation of I_1 on $S_r^{a,b}$.

LEMMA 2. *For every* $\epsilon > 0$, *there exists* $\delta_0 > 0$ *such that, for all* $\delta \leqslant \delta_0$ *we have*

$$\sup_{(u_1,u_2)\in S_r^{a,b}} |I_1(u_1, u_2) - I_2(u_1, u_2)| < \epsilon.$$

Proof. By using mainly the Lebourg's mean value theorem for locally Lipschitz functionals (see [3]) and the hypothesis (H_5) we find

$$|G(u)| \leqslant \|g(\cdot,0)\|_{L^1} + \|\theta_1\|_{L^{\frac{p}{p-1}}} \cdot \|u\|_{L^p} + \delta \|u\|_{L^p}^p.$$

Taking into account the hypothesis (H_6) and the fact that $(u_1, u_2) \in S_r^{a,b}$ we derive that

$$\begin{aligned} |I_1(u_1,u_2) - I_2(u_1,u_2)| =& |G(u_1-u_2) + <\phi, u_1>_V + <\phi, u_2>_V| \leqslant \\ \leqslant & \|g(\cdot,0)\|_{L^1} + \|\theta_1\|_{L^{\frac{p}{p-1}}} \cdot C_p(\Omega) \cdot r \\ & \cdot \left(\frac{1}{\sqrt{ab_1}} + \frac{1}{\sqrt{bb_2}}\right) + \delta \cdot C_p^p(\Omega) \\ & \cdot r^p \left(\frac{1}{\sqrt{ab_1}} + \frac{1}{\sqrt{bb_2}}\right)^p + \delta \cdot r \\ & \cdot \left(\frac{1}{\sqrt{ab_1}} + \frac{1}{\sqrt{bb_2}}\right) < \epsilon, \end{aligned}$$

for $\delta > 0$ small enough. □

LEMMA 3. *The functional I_2 satisfies the Palais-Smale condition on $S_r^{a,b}$.*

Proof. For the beginning it is important to remark that the expression of the generalized gradient $\partial(I_{2|S_r^{a,b}})$ at the point $(u_1, u_2) \in S_r^{a,b}$ is given by the formula

$$\partial(I_{2|S_r^{a,b}})(u_1,u_2) = \{\xi - r^{-2}\langle \xi, (u_1,u_2)\rangle_{V\times V} \cdot \Lambda(u_1,u_2) \ : \ \xi \in \partial I_2(u_1,u_2)\},$$

where $\Lambda : V \times V \to V^* \times V^*$ is the appropriate duality map given in (4). Here, the duality $\langle \cdot,\cdot \rangle_{V\times V}$ is taken for the norm

$$\|(u_1,u_2)\|_{V\times V} := \sqrt{a(B_1u_1,u_1)_V + b(B_2u_2,u_2)_V}, \ \forall\, u_1,u_2 \in V.$$

Let us consider a sequence $(u_n^1, u_n^2) \subset S_r^{a,b}$ such that

$$\sup_n |(I_{2_{|S_r^{a,b}}})(u_n^1,u_n^2)| < +\infty$$

and such that there exists some sequence $J_n \subset V^* \times V^*$ fulfilling the conditions

$$J_n \in \partial I_2(u_n^1,u_n^2), \quad n \geqslant 1$$

and

$$J_n - r^{-2}\left\langle J_n, (u_n^1,u_n^2)\right\rangle_{V\times V} \cdot \Lambda(u_n^1,u_n^2) \to 0, \tag{15}$$

strongly in $V^* \times V^*$. For concluding it suffices to prove that $\{(u_n^1, u_n^2)\}$ contains a convergent subsequence in $V \times V$. Under hypothesis (H_1) the functionals J and

G are Lipschitz continuous on bounded sets in $L^p(\Omega;\mathbf{R}^N)$ and their generalized gradients satisfy (cf. Clarke [3], Theorem 2.7.5)

$$\partial J(v) \subset \int_\Omega \partial_y j(x, v(x))dx$$

and

$$\partial G(v) \subset \int_\Omega \partial_y g(x, v(x))dx,\ \forall\, v \in L^p(\Omega;\mathbf{R}^N).$$

The density of V into $L^p(\Omega;\mathbf{R}^N)$ allows us to apply Theorem 2.2 of Chang [2]. Thus, we obtain

$$\partial(J_{|V})(v) \subset \partial J(v),$$

and

$$\partial(G_{|V})(v) \subset \partial G(v),\ \forall\, v \in V.$$

From $J_n \in \partial I_2(u_n^1, u_n^2)$ we derive that there exists $z_n^i \in \partial f_i(u_n^i)(i = 1, 2)$, $w_n \in \partial(J_{|V})(u_n^1 - u_n^2)$ and $z_n \in \partial(G_{|V})(u_n^1 - u_n^2)$ such that

$$J_n = (a_1(u_n^1, \cdot) + z_n^1 + \phi, a_2(u_n^2, \cdot) + z_n^2 + \phi) + K^*(w_n) + K^*(z_n),$$

where $K : V \times V \to V$ is the map given by

$$K(v_1, v_2) = v_1 - v_2.$$

By the above considerations we have that

$$w_n(x) \in \partial_y j(x, (u_n^1 - u_n^2)(x))$$

and

$$z_n(x) \in \partial_y g(x, (u_n^1 - u_n^2)(x)),\ \text{for a.e. } x \in \Omega.$$

By the relation (15) we get

$$\begin{aligned}&\big(a_1(u_n^1, \cdot) + z_n^1 + \phi, a_2(u_n^2, \cdot) + z_n^2 + \phi\big) + K^*(w_n) + K^*(z_n) - \\ &\quad - r^{-2}\langle[(a_1(u_n^1, \cdot) + z_n^1 + \phi, a_2(u_n^2, \cdot) + z_n^2 + \phi) + K^*(w_n) + K^*(z_n)],\end{aligned}$$

$$(u_n^1, u_n^2)\rangle_{V\times V} \cdot \Lambda(u_n^1, u_n^2) \to 0,\ \ \text{strongly in } V^* \times V^*.$$

Since the sequence (u_n^1, u_n^2) is contained in $S_r^{a,b}$ and by the coercivity property of B_1 and B_2 it follows that each sequence (u_n^1) and (u_n^2) is bounded in V. So, up to a subsequence, we may conclude that

$$u_n^i \rightharpoonup u_i,\ \text{weakly in } V,\ \text{for some } u_i \in V, (i = 1, 2).$$

The compactness assumptions in the hypothesis (H_2) implies that, again up to a subsequence,

$$z_n^i \to z_i, \text{ strongly in } V^*, \text{ for some } z_i \in V^* \ (i = 1, 2).$$

Also we have

$$\begin{aligned} w_n &\in \partial(J_{|V})(u_n^1 - u_n^2) \subset \partial J(u_n^1 - u_n^2), \\ z_n &\in \partial(G_{|V})(u_n^1 - u_n^2) \subset \partial G(u_n^1 - u_n^2). \end{aligned} \tag{16}$$

The compactness of the embedding $V \subset L^p(\Omega; \mathbf{R}^N)$ provides the convergences

$$u_n^i \to u_i, \text{ strongly in } L^p(\Omega; \mathbf{R}^N), \ (i = 1, 2). \tag{17}$$

Since J and G are locally Lipschitz on $L^p(\Omega; \mathbf{R}^N)$, the above property ensures that (w_n) and (z_n) are bounded in $L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$. By the reflexivity of $L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ and the compactness of the embedding $L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N) \subset V^*$, there exist $w, z \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ such that, up to a subsequence,

$$w_n \to w \text{ strongly in } V^* \text{ and weakly in } L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$$

and

$$z_n \to z \text{ strongly in } V^* \text{ and weakly in } L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N).$$

Proposition 2.1.5 in Clarke [3] and the relations (16) and (17) yield

$$\begin{aligned} w &\in \partial J(u_1 - u_2), \\ z &\in \partial G(u_1 - u_2). \end{aligned} \tag{18}$$

With the above remarks we may suppose that

$$a_i(u_n^i, u_n^i) \text{ converges in } \mathbf{R}, \ i = 1, 2,$$

and

$$\left\langle \left[(z_n^1 + \phi, z_n^2 + \phi) + K^*(w_n) + K^*(z_n) \right], (u_n^1, u_n^2) \right\rangle_{V \times V}$$

possesses a convergent subsequence in $\mathbf{R}$. From (15) and taking into account the convergences stated above we derive that

$$\left(a_1(u_n^1, \cdot), a_2(u_n^2, \cdot) \right) - \lambda_0 \cdot \Lambda(u_n^1, u_n^2),$$

converges strongly in $V^* \times V^*$, where λ_0 is the one required in (H_7). So, hypothesis (H_7) allows us to conclude that (u_n^1, u_n^2) has a convergent subsequence in $V \times V$, so in $S_r^{a,b}$. Thus the Palais-Smale condition for the functional I_2 on $S_r^{a,b}$ is satisfied and the proof is now complete. □

LEMMA 4. *If* $u = (u_1, u_2)$ *is a critical point of* $I_{2_{|S_r^{a,b}}}$ *then there exists a pair* $(\lambda_1, \lambda_2) \subset \mathbf{R}^2$ *such that* $((u_1, u_2), (\lambda_1, \lambda_2))$ *is a solution of the problem* $(P^2_{r,a,b})$.

Proof. Since u is a critical point for $I_{2_{|S_r^{a,b}}}$, it follows that

$$0_{V\times V} \in \left(\partial I_{2_{|S_r^{a,b}}}\right)(u_1, u_2) \tag{19}$$

Taking into account the expression of the generalized gradient of the restriction of I_2 at $S_r^{a,b}$, we may conclude the existence of an element $\xi \in \partial I_2(u_1, u_2)$ such that

$$\xi - r^{-2}\langle \xi, (u_1, u_2)\rangle_{V\times V} \cdot \Lambda(u_1, u_2) = 0 \tag{20}$$

By the Clarke's calculus and the inclusions stated in the proof of Lemma 3 we derive

$$\begin{aligned}
&\partial I_2(u_1, u_2)(v_1, v_2) \subset a_1(u_1, v_1) + \\
&\quad + a_2(u_2, v_2) + \partial f_1(u_1)v_1 + \partial f_2(u_2)v_2 \\
&\quad + \int_\Omega \partial_y j(x, (u_1 - u_2)(x))(v_1 - v_2)(x)dx + \\
&\quad + \int_\Omega \partial_y g(x, (u_1 - u_2)(x))(v_1 - v_2)(x)dx \\
&\quad + < \phi, v_1 >_V + < \phi, v_2 >_V,
\end{aligned}$$

for all $v_1, v_2 \in V$. So, there exists some $z_i \in \partial f_i(u_i)$ $(i = 1, 2)$ and $w, z \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ with

$$w(x) \in \partial_y j(x, (u_1 - u_2)(x)) \ \text{ for a.e. } x \in \Omega,$$

and

$$z(x) \in \partial_y g(x, (u_1 - u_2)(x)) \ \text{ for a.e. } x \in \Omega,$$

such that

$$\begin{aligned}
\langle \xi, (v_1, v_2)\rangle_{V\times V} = &\, a_1(u_1, v_1) + a_2(u_2, v_2) + < z_1, v_1 >_V \\
&+ < z_2, v_2 >_V + \int_\Omega < w(x), (v_1 - v_2)(x) > dx \\
&+ \int_\Omega < z(x), (v_1 - v_2)(x) > dx \\
&+ < \phi, v_1 >_V + < \phi, v_2 >_V .
\end{aligned}$$

From (20) it follows that

$$
\begin{aligned}
&a_1(u_1, v_1) + a_2(u_2, v_2) + < z_1, v_1 >_V + < z_2, v_2 >_V \\
&\quad + \int_\Omega < w(x), (v_1 - v_2)(x) > dx \\
&\quad + \int_\Omega < z(x), (v_1 - v_2)(x) > dx \\
&\quad + < \phi, v_1 >_V + < \phi, v_2 >_V \\
&\quad - r^{-2}[a_1(u_1, u_1) + a_2(u_2, u_2) + < z_1, u_1 >_V + < z_2, u_2 >_V \\
&\quad + \int_\Omega < w(x), (u_1 - u_2)(x) > dx \\
&\quad + \int_\Omega < z(x), (u_1 - u_2)(x) > dx \\
&\quad + < \phi, u_1 >_V + < \phi, u_2 >_V] \cdot (a(B_1 u_1, v_1)_V \\
&\quad + b(B_2 u_2, v_2)_V) = 0,
\end{aligned}
$$

for all $v_1, v_2 \in V$. Set

$$
\begin{aligned}
\lambda = & r^{-2}[a_1(u_1, u_1) + a_2(u_2, u_2) + < z_1, u_1 >_V + < z_2, u_2 >_V \\
&+ \int_\Omega < (w + z)(x), (u_1 - u_2)(x) > dx + < \phi, u_1 >_V + < \phi, u_2 >_V].
\end{aligned}
$$

Let us now observe that we have

$$
\begin{aligned}
&\int_\Omega \langle (w + z)(x), (v_1 - v_2)(x) \rangle dx \\
&\quad \leqslant \int_\Omega \max\{\langle \mu_1, (v_1 - v_2)(x) \rangle; \mu_1 \in \partial_y j(x, (u_1 - u_2)(x))\} \\
&\quad + \int_\Omega \max\{\langle \mu_2, (v_1 - v_2)(x) \rangle; \mu_2 \in \partial_y g(x, (u_1 - u_2)(x))\} \\
&\quad = \int_\Omega j_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x)) dx \\
&\quad + \int_\Omega g_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x)) dx.
\end{aligned}
$$

In the above relation, the last equality holds because of Proposition 2.1.2 in [3]. Taking into account the choice of $z_i (i = 1, 2)$, z and w, it is easily to observe that if we denote $\lambda_1 = \lambda a$ and $\lambda_2 = \lambda b$, our hypothesis (H_2) and some simple calculation lead us to the desired conclusion claimed in the formulation of Lemma 4. □

4. The main result

With the preliminary results stated in Section 3 we are now prepared to prove our perturbation result.

THEOREM 1. *Assume that the hypotheses* $(H_1) - (H_7)$ *are fulfilled. Then, for every* $n \geqslant 1$, *there exists* $\delta_n > 0$ *such that, for each* $\delta \leqslant \delta_n$, *the problem* $(P^2_{r,a,b})$ *admits at least n distinct solutions.*

Proof. Fix $n \geqslant 1$. By Lemma 4 it suffices to prove the existence of a $\delta_n > 0$ such that, for every $\delta \leqslant \delta_n$, the functional $I_{2_{|S_r^{a,b}}}$ has at least n distinct critical values. We may use now the conclusion of Lemma 1 and this implies that it is possible to consider a sequence (b_n) of essential values of $I_{1_{|S_r^{a,b}}}$, strictly increasing to s. Choose an arbitrary $\epsilon_0 < \frac{1}{2}\min_{1\leqslant i\leqslant n-1}(b_{i+1} - b_i)$. We now apply Theorem 2.6 from [5] to the functionals $I_{1_{|S_r^{a,b}}}$ and $I_{2_{|S_r^{a,b}}}$. Thus, for every $1 \leqslant i \leqslant n-1$, there exists $\eta_i > 0$ such that the relation

$$\sup_{(u_1,u_2)\in S_r^{a,b}} |I_1(u_1, u_2) - I_2(u_1, u_2)| < \eta_i$$

implies the existence of an essential value c_i of $I_{2_{|S_r^{a,b}}}$ in $(b_i - \epsilon_0, b_i + \epsilon_0)$. By taking $\epsilon = \min\{\epsilon_0, \eta_1, \cdots, \eta_{n-1}\}$ in Lemma 2, we derive the existence of a $\delta_n > 0$ such that

$$\sup_{(u_1,u_2)\in S_r^{a,b}} |I_1(u_1, u_2) - I_2(u_1, u_2)| < \epsilon,$$

provided $\delta \leqslant \delta_n$ in (H_5) and (H_6). So, the functional $I_{2_{|S_r^{a,b}}}$ has at least n distinct essential values $c_1, c_2, \cdots, c_n$ in $(-\infty, b_n + \epsilon)$. For concluding our proof it suffices to show that $c_1 \cdots, c_n$ are critical values of $I_{2_{|S_r^{a,b}}}$. The first step is to prove that there exists $\epsilon > 0$ such that $I_{2_{|S_r^{a,b}}}$ has no critical value in $(c_i - \epsilon, c_i + \epsilon)$. Indeed, if this is not the case, there exists a sequence (d_n) of critical values of $I_{2_{|S_r^{a,b}}}$ with $d_n \to c_i$ as $n \to \infty$. The fact that d_n are critical values for the restriction of I_2 at $S_r^{a,b}$ implies that for every $n \geqslant 1$, there exists $(u_n^1, u_n^2) \in S_r^{a,b}$ such that

$$I_2(u_n^1, u_n^2) = d_n \text{ and } \lambda^*(u_n^1, u_n^2) = 0,$$

where λ^* is the lower semicontinuous functional defined by

$$\lambda^*(u_1, u_2) := \min\{\|(\xi_1, \xi_2)\|_{V^*\times V^*}; (\xi_1, \xi_2) \in \partial I_{2|S_r^{a,b}}(u_1, u_2)\}.$$

Thus, passing eventually to a subsequence, $(u_n^1, u_n^2) \to (u_1, u_2) \in S_r^{a,b}$, strongly in $V \times V$. The continuity of I_2 and the lower semicontinuity of λ^* implies that

$$I_2(u_1, u_2) = c_i \text{ and } \lambda^*(u_1, u_2) = 0,$$

which contradicts the initial conditions on c_i. Let us fix $c_i - \epsilon < a < b < c_i + \epsilon$. By Lemma 3, I_2 satisfies the Palais-Smale condition on $S_r^{a,b}$. So, for every point $e \in [a, b]$, $(PS)_e$ holds. We have fulfilled the set of conditions which allow us to apply the 'Noncritical Interval Theorem' due to J.- N. Corvellec, M. Degiovanni and M. Marzocchi (see Theorem 2.15 in [4]), on the complete metric space $\left(S_r^{a,b}, d(\cdot,\cdot)\right)$, where by $d(\cdot,\cdot)$ we have denoted the geodesic distance on $S_r^{a,b}$, that is, for every points $x, y \in S_r^{a,b}$, $d(x, y)$ is equal to the infimum of the lengths of all paths on $S_r^{a,b}$ joining x and y. We obtain that there exists a continuous map $\eta : S_r^{a,b} \times [0, 1] \to S_r^{a,b}$ such that, for each $(u = (u_1, u_2), t) \in S_r^{a,b} \times [0, 1]$, are satisfied the conditions

(a) $\eta(u, 0) = u$,

(b) $I_2(\eta(u, t)) \leqslant I_2(u)$,

(c) $I_2(u) \leqslant b \Longrightarrow I_2(\eta(u, 1)) \leqslant a$,

(d) $I_2(u) \leqslant a \Longrightarrow \eta(u, t) = u$.

By the above conditions, it follows that the map

$$[I_{2_{|S_r^{a,b}}} \leqslant b] \ni u \mapsto \eta(u, 1) \in [I_{2_{|S_r^{a,b}}} \leqslant b]$$

is a retraction. Let us define the map $\Psi : [I_{2_{|S_r^{a,b}}} \leqslant b] \times [0, 1] \to [I_{2_{|S_r^{a,b}}} \leqslant b]$ by the relation

$$\Psi(u, t) = \eta(u, t).$$

Since for every $u \in [I_{2_{|S_r^{a,b}}} \leqslant b]$, we have

$$\Psi(u, 0) = u, \;\; \Psi(u, 1) = \eta(u, 1),$$

and for each $(u, t) \in [I_{2_{|S_r^{a,b}}} \leqslant b] \times [0, 1]$, the equality $\Psi(u, t) = \Psi(u, 0)$ holds, it follows that Ψ is $[I_{2_{|S_r^{a,b}}} \leqslant b]-$ homotopic to the identity of $[I_{2_{|S_r^{a,b}}} \leqslant b]$. Thus, Ψ is a strong deformation retraction which implies that the pair

$$\left([I_{2_{|S_r^{a,b}}} \leqslant b], [I_{2_{|S_r^{a,b}}} \leqslant a]\right)$$

is trivial. With this argument, we get that c_i is not an essential value of the restriction of I_2 at $S_r^{a,b}$. This is the contradiction which concludes our proof. □

5. Applications

In many problems arising in Mechanics and Engineering the cost or the weight of the structure may be expressed as a linear function of the norm of the unknown function. Thus the constraint that we have imposed $\|u\|_V = r$(or, equivalently, $a\|u_1\|^2 + b\|u_2\|^2 = r^2$) means that we have a system with prescribed cost or weight, or in some cases energy consumption. The stability analysis of such a system involving nonconvex nonsmooth potential functions (called also nonconvex superpotential) leads to the treatment of a double eigenvalue problem for hemivariational inequalities on a spherelike manifold. We begin with two mathematical examples and then we shall give some applications from Mechanics.

5.1. PERTURBATIONS OF A COUPLED SEMILINEAR POISSON EQUATION

First, we consider the case of the problem $(P^1_{r,a,b})$ in which $C \equiv 0$, $B_1 = B_2 = id_V$, $a = b = 1$. Moreover a_1, a_2 are coercive, in the sense that

$$a_i(v, v) \geqslant \bar{a}_i \|v\|^2_V, \ \forall v \in V, i = 1, 2,$$

for some constants $\bar{a}_1, \bar{a}_2 > 0$ and $j : \mathbf{R} \to \mathbf{R}$ is the primitive

$$j(t) = \int_0^t \varphi(\tau) d\tau, \ t \in \mathbf{R},$$

with $\varphi : \mathbf{R} \to \mathbf{R}$ even, locally bounded, measurable and satisfying the subcritical growth condition : for some $1 \leqslant p < \frac{2m}{m-2}$, if $m \geqslant 3$ ($1 \leqslant p < +\infty$, if $m = 1, 2$), we have

$$|\varphi(t)| \leqslant c_1 + c_2 |t|^{p-1}, \ \forall t \in \mathbf{R}.$$

It is known that

$$\partial j(t) \subset [\underline{\varphi}(t), \overline{\varphi}(t)], \ \forall t \in \mathbf{R},$$

where

$$\underline{\varphi}(t) = \lim_{\delta \to 0} \operatorname{essinf} \{\varphi(s) \ ; \ |t - s| < \delta\}$$

and

$$\overline{\varphi}(t) = \lim_{\delta \to 0} \operatorname{esssup} \{\varphi(s) \ ; \ |t - s| < \delta\}$$

(see [2]). Suppose further the sign condition of Chang [2]

$$\underline{\varphi}(t) > 0 \text{ if } t < 0 \text{ and } \overline{\varphi}(t) < 0 \text{ if } t > 0.$$

Let us consider that the superpotential j gives rise to a very irregular graph $[\xi, \partial j(\xi)]$ (i.e. the graph of ∂j has many zig - zag etc.). Then we consider the eigenvalue problem $(P^2_{r,a,b})$, where g^0_y is appropriately chosen in order to "smoother a little bit" the graph $[\xi, \partial j(\xi)]$, i.e. the graph $[\xi, \partial j(\xi) + \partial g(\xi)]$ has a smaller number of irregularities than the graph $[\xi, \partial j(\xi)]$. In the present case we may consider that

$$\partial j(t) + \partial g(t) \subset [\underline{\varphi}(t) + \underline{\varphi^1}(t), \overline{\varphi}(t) + \overline{\varphi^1}(t)], \ \forall t \in \mathbf{R}.$$

In fact, we consider

$$g(t) = \int_0^t \varphi^1(\tau)d\tau,\ t \in \mathbf{R},$$

where $\varphi^1 : \mathbf{R} \to \mathbf{R}$ is locally bounded, measurable and satisfies the subcritical growth condition

$$|\varphi^1(t)| \leqslant c_3 + c_4|t|^{p-1},\ \forall t \in \mathbf{R}$$

Note that we do not need to impose to φ^1 that it is even, as we have assumed on φ. Obviously, Theorem 1 applies on every sphere $\|v_1\|_V^2 + \|v_2\|_V^2 = r^2$ of $V \times V$, with a sufficiently small $r > 0$. More precisely, for every $n \geqslant 1$, there exists $\delta_n > 0$ such that if c_3 and c_4 are chosen smaller than δ_n, then the perturbed problem $(P^2_{r,a,b})$ admits at least n distinct solutions.

As a specific example of application of Theorem 1, we consider the coupled semilinear Poisson equations on a bounded domain Ω in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$ in the double eigenvalue problem

$$\triangle u_1 + \lambda_1 u_1 \in [\underline{\varphi}(u_1(x) - u_2(x)), \overline{\varphi}(u_1(x) - u_2(x))] \text{ for a.e. } x \in \Omega$$

$$\triangle u_2 + \lambda_2 u_2 \in [-\overline{\varphi}(u_1(x) - u_2(x)), -\underline{\varphi}(u_1(x) - u_2(x))] \text{ for a.e. } x \in \Omega$$

$$u_1 = u_2 = 0 \text{ on } \partial\Omega.$$

Here $\lambda_1, \lambda_2 \in \mathbf{R}$ are the eigenvalues, u_1, u_2 are the corresponding eigenfunctions and $\underline{\varphi}, \overline{\varphi}$ are determined above for the function $\varphi : \mathbf{R} \to \mathbf{R}$. We choose $V = H_0^1(\Omega)$,

$$a_1(u, v) = a_2(u, v) = \int_\Omega \nabla u \cdot \nabla v dx,\ \forall u, v \in H_0^1(\Omega),$$

$$(B_1 u, v)_{H_0^1} = (B_2 u, v)_{H_0^1} = \int_\Omega u \cdot v dx,\ \forall u, v \in H_0^1(\Omega),$$

$j : \mathbf{R} \to \mathbf{R}$ being equal to the primitive of φ as we considered above and, for simplicity, $C \equiv 0$. Notice that each eigensolution of the hemivariational inequality appearing in the problem $(P^1_{r,a,b})$ represents a weak solution of the Dirichlet system above. Under the growth condition for φ as above and the assumptions from the section 2 on j, Theorem 1 in [1] implies the existence of infinitely many double eigenfunctions $(u_n^1, u_n^2) \in S_r^{a,b}$, with $u_n^1, u_n^2 \in H_0^1(\Omega) \cap H^2(\Omega)$ for the foregoing Dirichlet problem.

Further, we consider the perturbed eigenvalue problem

$$\triangle u_1 + \lambda_1 u_1 \in [\underline{\varphi}(u_1(x) - u_2(x)) + \underline{\varphi^1}(u_1(x) - u_2(x)),$$

$$\overline{\varphi}(u_1(x) - u_2(x)) + \overline{\varphi^1}(u_1(x) - u_2(x))] \text{ for a.e. } x \in \Omega$$

$$\triangle u_2 + \lambda_2 u_2 \in [-\overline{\varphi}(u_1(x) - u_2(x)) + \underline{\varphi}^1(u_1(x) - u_2(x)),$$

$$-\underline{\varphi}(u_1(x) - u_2(x)) + \overline{\varphi^1}(u_1(x) - u_2(x))] \text{ for a.e. } x \in \Omega$$

$$u_1 = u_2 = 0 \text{ on } \partial\Omega,$$

where φ^1 is chosen as in the previous example and satisfies the conditions therein. Then, our Theorem 1 applies and we obtain that the perturbed Dirichlet problem considered above admits infinitely many distinct solutions. Notice that c_3 and c_4 must be sufficiently small, in the same sense as in the first case considered in this section.

5.2. ADHESIVELY CONNECTED VON KÁRMÁN PLATES: BUCKLING FOR GIVEN COST OR WEIGHT

In the framework of the theory of elastic von Kármán plates, i.e. of plates having large deflections, we consider two or more such plates connected with an adhesive material. The behaviour of the adhesive material may be described by a relation of the form

$$-f \in \partial j(u_1 - u_2), \tag{21}$$

(cf. Panagiotopoulos [22], p. 109).The graph of $\{f, u_1 - u_2\}$ may be a zig-zag graph with complete vertical branches in the most general case. Concerning the derivation and study of the corresponding hemivariational inequalities we refer to [16], [22]. We assume that we have two plates Ω_1 and $\Omega_2, \Omega_i \subset \mathbf{R}^2, i = 1, 2,$ which are adhesively connected on $\Omega' \subset \Omega_i, i = 1, 2$. The plates have the boundaries Γ_1 and Γ_2 respectively and $\overline{\Omega} \cap \Gamma_i = \emptyset, i = 1, 2$. The boundaries are assumed to be Lipschitzian and are not subjected to any loading on Ω_1 and Ω_2 vertical to the middle plate plane or parallel to it. We assume that $\Omega_1 \equiv \Omega_2$ as subsets of $\mathbf{R}^2$ and we denote both Ω_1 and Ω_2 by Ω. The plates are only subjected along their boundaries Γ_1 and Γ_2 to continuously distributed compressive forces, i.e.

$$\sigma_{\alpha\beta i} n_{\alpha_i} = \lambda_i g_{\alpha_i} \quad \alpha, \beta = 1, 2, \ i = 1, 2,$$

where $\sigma = \{\sigma_{\alpha\beta}\}$ denotes the stress tensor for the in-plane action of the plate, $n = \{n_\alpha\}$ is the outher unit normal vector to Γ_1 or to Γ_2, $g_i = \{g_{1_i}, g_{2_i}\}$ is a given force distribution, which is self equilibrated, i.e. for each plate

$$\int_{\Gamma_i} g_{\alpha_i} ds = 0, \quad \int_{\Gamma_i} (x_1 g_{2_i} - g_{1_i} x_2) ds = 0, \quad i = 1, 2.$$

Here $\lambda_i, i = 1, 2,$ is a real number which measures the magnitude of the compressive forces having the direction $g_i, i = 1, 2,$ along the boundaries of the plates. These compressive forces may cause buckling of the composite plate with partial debonding of the adhesive material. As in [15], p. 455 and in [21] p. 234, where the analogous buckling problem for variational inequalities is formulated, the notion of 'reduced variational solution' is introduced and we obtain the following eigenvalue problem: Find $u_1, u_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbf{R}$ such that

$$
\begin{aligned}
&a_1(u_1, v_1) + a_2(u_2, v_2) + (C_1(u_1), v_1)_V + (C_2(u_2), v_2)_V \\
&\quad + \int_\Omega j_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x))dx \\
&\quad \geqslant \lambda_1 (B_1 u_1, v_1)_V + \lambda_2 (B_2 u_2, v_2)_V ,
\end{aligned}
$$

for all $v_1, v_2 \in V$. Here V is the real Sobolev space $H^2(\Omega)$ with inner product $(\cdot, \cdot)_V$, $a_i(u_i, v_i)$ is the bending energy of the plate i, $(C_i(u_i), v_i)$, with $C_i(\cdot)$ a nonlinear compact operator, is the bending energy of the plate i due to the stretching of the same plate, $j^0(x, u_1 - u_2; v_1 - v_2)$ denotes the directional derivative in the sense of Clarke at the state $(u_1 - u_2)(x)$ and in the direction $(v_1 - v_2)(x)$ at x, and $(B_i u_i, v_i)$ is given by the relation (7.2.13) of [21], i.e.

$$
(B_i u_i, v_i) = - > \int_{\Omega_i} h_i \sigma^0_{\alpha\beta_i} u_{i,\alpha} v_{i,\beta} dx \; \forall \; u_i, v_i \in V,
$$

for $i = 1, 2$. Here h_i denotes the thickness of the plate i and $\alpha\beta_i^0$ the stress field in the plane of the plate i caused by the forces $g_{\alpha_i}(\alpha, \beta = 1, 2, \;\; i = 1, 2)$. Moreover we note that on Γ_i, concerning the plate bending, boundary conditions which guarantee the coercivity of the bilinear forms $a_i(\cdot, \cdot), i = 1, 2,$ are assumed to hold. For instance the built-in boundary conditions $u_i = \frac{\partial u_i}{\partial n} = 0, i = 1, 2,$ or the simple support boundary conditions $u_i = 0, M_i(u_i) = 0, i = 1, 2,$ where M_i denotes the bending element of the $i-$th plate. Further we shall not need for the operators B_i the property that $(B_i u_i, v_i) > 0 \; \forall \; u_i \in V, u_i \neq 0,$ as it is the case in the corresponding theory (see Naumann and Wenk [15]) of eigenvalue problems for variational inequalities but the stronger property of coercivity (this property is a consequence of the assumption that the stress vector on the boundary of each subdomain Ω_{0_i} of $\Omega, i = 1, 2,$ is directed outside of Ω_{0_i}, i.e. that each subdomain of the plate is subjected to compressive forces, (cf. Naumann and Wenk [15], p. 457)). Further we express the total cost or weight of the structure by the form $\sum_{i=1}^2 a_i \, (B_i u_i, v_i) = r^2$, where a_i are given positive constants. We get that for the arising double eigenvalue problem for hemivariational inequalities $(P^1_{r,a,b})$ the hypotheses are satisfied and the multiplicity result of Theorem 1 in [1] holds.

5.2.1. *Perturbations of the buckling problem of a sandwich beam of prescribed weight*

Let us now consider the perturbed hemivariational inequality : Find $u_1, u_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbf{R}$ such that

$$\begin{aligned} &a_1(u_1, v_1) + a_2(u_2, v_2) + (C_1(u_1), v_1)_V + (C_2(u_2), v_2)_V \\ &\quad + \int_\Omega \{ j_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x)) \\ &\quad + g_y^0(x, (u_1 - u_2)(x); (v_1 - v_2)(x))\} dx \\ &\quad \geqslant \lambda_1 \, (B_1 u_1, v_1)_V + \lambda_2 \, (B_2 u_2, v_2)_V \, , \end{aligned}$$

for all $v_1, v_2 \in V$. One can assume that the graph $[\xi, \partial j(\xi) + \partial g(\xi)]$ is much more regular than the graph $[\xi, \partial j(\xi)]$. Further one can assume that the graph $[\xi, \partial j(\xi) + \partial g(\xi)]$ is monotone, a fact which in the framework of a numerical calculation is beneficial. Moreover, in the monotone case one can consider the corresponding variational inequality - eigenvalue problem and get some useful comparison results (especially in the case of simple eigenvalue problems for which there exist certain results for variational inequalities (see Le and Schmitt [7]).

5.2.2. *Fuzzy effects superimposed on an adhesive contact law*

Let us put ourselves in the framework of the previous example of adhesively connected plates and let us consider the following interface law (see Panagiotopoulos [22], p. 77)

$$-f(x) \in \partial j([u](x)) + \partial g(u(x)), \tag{22}$$

where ∂g describes the fuzzy effects. We recall that g results in the following manner (see Rockafellar [25])

Let l be an open subset of the real line $\mathbf{R}$ and let M be a measurable subset of l such that for every open and nonempty subset I of l, $\text{mes}(I \cap (l - M))$ is > 0. Let

$$r(u(x)) = \begin{cases} +b_1 & \text{if } u(x) \in M \\ -b_2 & \text{if } u(x) \notin M \end{cases}$$

and $g(u) = \int_0^u r(u^*) du^*$. Then g is Lipschitzian and

$$\partial g(u) = [-b_2, b_1], \; \forall u(x) \in l.$$

Thus $\partial g(u(\cdot))$ has an infinite number of jumps in l where each jump is identified with the interval $[-b_2, b_1]$. In the composite law (22), the zero of this interval lies on the graph of $[\xi, j(\xi)]$ and the zone $[-b_2, b_1]$ around this graph describes the fuzzy nature of the adhesive contact law. Note that existence results related to fuzzy effects have been studied by Naniewicz and Panagiotopoulos in [14] p. 132. Here we can apply our results to the perturbed problem $(P^2_{r,a,b})$, i.e. to the system related to the interface law (22). Our Lemma 2 shows that if the fuzzy effect tends to disappear then the energy of the perturbed problem tends to the energy of the initial nonfuzzy problem. On the other hand, by Theorem 1, the number of solutions of the perturbed problem tends to infinity if the perturbation given by the fuzzy effect tends to zero. We also remark that our results hold if the fuzzy effect is linked to a subcritical growth, but is arbitrary, in the sense that it has no symmetry.

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